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# EFFECT OF THE HISTORY TERM ON THE MOTION OF RIGID SPHERES IN A VISCOUS FLUID

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**Abstract**—The inclusion of the history term (often called "the Basset term") in the equation of motion of a sphere makes this equation non-explicit in the velocity or acceleration. For this reason, the numerical solution of the equation becomes cumbersome and computationally time-consuming. By the use of an integrodifferential operator, the equation of motion of a sphere is transformed to a second-order ordinary differential equation, which is explicit in the velocity. This equation is solved numerically to determine the effect of the history term on the calculations of particle velocity and trajectories in unsteady flows. The numerical calculations make use of empirical correction factors to account for the effects of finite Reynolds number. Computations are made with the fluid velocity being sinusoidal, random and monotonically increasing. It was observed that the effect of the history term was more pronounced at higher frequencies of the fluid velocity and for fluid-to-particle density ratios higher than 0.002.

*Key Words:* history term, Basset force, particle motion, unsteady flow

## INTRODUCTION

The last 10 years have seen a dramatic increase in the number of studies which involve the Lagrangian dynamic simulation of particles and bubbles. The wide and ever increasing availability of computational power has made it possible for single or ensembles of spheres† to be tracked in unsteady flow fields. Although an exact form of the equation of motion of the particles is available (Maxey & Riley 1983), researchers are generally using simplified versions of it (Clift *et al.* 1978) which best suit their applications. Using the exact form of the equation of motion of the particles has two drawbacks: first, it is only applicable to very low Reynolds numbers; and second, it contains terms which are time-consuming when calculated repetitively.

To counteract the first drawback, researchers have resorted to using empirical coefficients for several of the terms in the equation of motion (Hjelmfelt & Mockros 1966; Odar & Hamilton 1964), most of which are summarized in Clift *et al.* (1978). These coefficients multiply the Stokes drag term ($c_1$), the added mass term ($\Delta_A$) and the history term, which is often called the Basset term ($\Delta_H$). The use of these empirical coefficients has enabled accurate calculations of particle trajectories in flow fields at high Reynolds numbers. These calculations have resulted in predictions of particle characteristics and behavior, such as local concentration, dispersion in a turbulent field or mass and momentum exchange with the carrier fluid.

Little progress has been made on the second drawback of the exact equation of motion (that certain terms, and especially the history term, are cumbersome to calculate in repetitive computations). The vast majority of the Lagrangian computations have been made for cases where the history term is very small in comparison to other terms. By using dimensional arguments this term is almost always neglected, a very convenient assumption which not only reduces the order of the differential equation of motion of the particle and makes it explicit in the velocity, but also diminishes the memory requirements of the computations by not retaining information on the history of the acceleration of the particle. Neglecting the history term does not change appreciably the calculations for particles of intermediate or large size. However, it was observed that for fine evaporating particles the history term accounts for approximately 20% of the total force (Li &

†To avoid unnecessary repetition, the term spheres will be used in general to include rigid particles, bubbles or droplets.

Michaelides 1992). Neglecting the history term therefore, may result in a substantial error in particle velocities and positions.

By using an integrodifferential transformation, Michaelides (1992) has transformed the equation of motion of the particle to a second-order differential equation. In this equation the history integral does not include the velocity of the particle. Therefore, the equation is explicit in the velocity of the particle and may be solved rather easily with a standard numerical technique. Another advantage of using the transformed equation is that it is no longer necessary to store past accelerations of the particle. This results in substantial computer memory reduction. It must be pointed out that this transformation still makes use of the past acceleration of the fluid itself, since the history term is transformed and not left out of the calculations.

The application of the transformed equation is extended in this paper to include the effects of finite Reynolds number and particle acceleration. A transformed equation is derived, which includes the drag, added mass and history coefficients. Subsequently, calculations are performed with the transformed equation to determine the effect of the history term on the trajectories of particles under various flow conditions. The final result of these calculations is the determination of cases in which the effects of the history term are important and, therefore, this term should not be neglected in the computations.

## THE EQUATION OF MOTION OF A PARTICLE

The equation of motion of a single particle in an unsteady flow is usually given in terms of the dimensionless relative velocity in the $i$th direction $w_i = v_i - u_i$ and reads as follows:

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} + \lambda c_1 w_i + \lambda \Delta_{\mathrm{H}} \sqrt{\frac{9\beta}{2\pi}} \left\{ \int_0^t \frac{\frac{\mathrm{d}w_i}{\mathrm{d}\sigma}}{\sqrt{t-\sigma}} \mathrm{d}\sigma + \frac{w_i(0^+)}{\sqrt{t}} \right\} = -\lambda(1-\beta)\frac{\mathrm{d}u_i}{\mathrm{d}t} + \lambda(1-\beta)G_i, \qquad [1]$$

where $\beta$ is the ratio of the fluid-to-particle density and $\lambda$ is a parameter which includes the added mass term; $\lambda = 1/(1 + \frac{1}{2}\Delta_{\mathrm{A}}\beta)$. Since $\Delta_{\mathrm{A}}$ is a function of the acceleration number (Odar & Hamilton 1964), $\lambda$ is not a constant. In [1], the velocity is made dimensionless by using the characteristic velocity of the fluid $U_0$ and time is non-dimensionalized by using the characteristic time of the particle $\tau_{\mathrm{P}} = 2\rho_{\mathrm{P}}\alpha^2/9\mu$. The first term on the left-hand side is the acceleration of the particle. The second term is the drag term with the empirical coefficient $c_1$. The third term represents the history of the particle as it moves in the unsteady flow field. The history term includes (the second part of the sum in the braces) the effect of a finite initial velocity of the particle $w_{i0}$. This part is often absent in commonly used expressions, where the assumption of zero initial velocity is made. It must be emphasized, however, that the exclusion of this part is tantamount to making the implicit assumption of zero initial relative velocity (Maxey 1987; Michaelides 1992). Of the terms on the right-hand side, the first represents the effect of the local acceleration of the fluid and the second the gravity/buoyancy effect. $G_i = g_i\tau_{\mathrm{P}}/U_0$, where $g_i$ is the gravitational acceleration, is the dimensionless gravity term. The Lagrangian derivative $\mathrm{d}/\mathrm{d}t$ is with respect to the moving particle. The drag, added mass and history term coefficients $c_1$, $\Delta_{\mathrm{A}}$ and $\Delta_{\mathrm{H}}$ are empirical expressions, which account for the finite Reynolds number effect. In the limit of very small particle Reynolds numbers their value is equal to 1.

The initial condition for this equation is $w_i(0) = w_{i0}$ and the initial acceleration is given by the following expression:

$$\frac{\mathrm{d}w_i}{\mathrm{d}t}(0) = \lambda c_1 w_{i0} - \lambda(1-\beta)\frac{\mathrm{d}u_i}{\mathrm{d}t}(0) + \lambda(1-\beta)G_i - \lambda\Delta_{\mathrm{H}}\sqrt{\frac{9\beta}{2\pi}} w_{i0}\delta(t), \qquad [2]$$

where $\delta(t)$ is the Dirac delta. The last term in [2] accounts for the impulse on the particle, which is introduced with finite velocity into a fluid.

Although [1] has been derived for rigid spheres at the limit of low Reynolds numbers (and strictly speaking it should only be applied to rigid spheres), it has been used extensively in calculations with viscous spheres (bubbles, droplets) or with non-spherical particles (Clift *et al.* 1978).

Regarding the history integral, it must be pointed out that recent results by Lawrence & Weinbaum (1988) and by Yang & Leal (1991) indicate that the kernel of the history integral as shown above is a special case applied only to rigid spheres. They have observed that when the sphere diverges from sphericity or perfect rigidity, the kernel of the history integral attains a more complicated form. Actually, in the case of a viscous sphere, Michaelides & Feng (1993) have shown that an explicit form (in the time domain) of the equation of motion can be obtained only in some special cases. Also, Mei *et al.* (1991) and Lovalenti & Brady (1993) allude to a different decay of the history integral (initially the decay is of the order of $t^{-1/2}$ and later of the order $t^{-2}$) when the inertia terms in the momentum equation of the fluid are considered. However, even with the different kernel or the faster decay in the latter stages of the motion, a history integral is always present.

Realistic Lagrangian calculations with the equation of motion at finite Reynolds numbers are always done with coefficients similar to those used in [1]. Clift *et al.* (1978) report such calculations for the acceleration of a particle from rest, where they show that the history term may account for as much as 20% of the acceleration at the early stages of the particle motion.

Equation [1] is not explicit in $w_i$ because of the history term. Hence, solving it numerically by any method implicit or explicit, involves time-consuming iterations. If the history term is neglected, the equation becomes explicit in the relative velocity and one may obtain particle velocities and trajectories easier and faster. It is believed that this is the main reason why the vast majority of repetitive computations (e.g. with Monte-Carlo simulations) have been conducted with the history term neglected, while other terms of apparently lesser magnitue (e.g. added mass) were retained. While it is correct to assume that the history term is negligible in comparison to the other terms, this is not always the case, expecially when the size of the particles is small and the fluid velocity varies at high frequency.

By the use of an integrodifferential transformation, [1] is converted into an explicit equation with respect to the particle relative velocity. This is accomplished by transforming the equation in the Laplace space, rearranging the resulting algebraic equation and transforming back into the time variable (Michaelides 1992). The resulting equation is a second-order differential equation and reads as follows:

$$\begin{aligned}\frac{d^2w_i}{dt^2}+\lambda\left(2c_1-\frac{9\beta\lambda\Delta_H^2}{2}\right)\frac{dw_i}{dt}+\lambda^2c_1^2w_i= &-\lambda(1-\beta)\frac{d^2u_i}{dt^2}-\lambda^2(1-\beta)c_1\frac{du_i}{dt}\\ &+\lambda^2(1-\beta)\Delta_H\sqrt{\frac{9\beta}{2\pi}}\int_0^t\frac{\frac{d^2u_i}{d\sigma^2}}{(t-\sigma)^{0.5}}d\sigma+\lambda\Delta_H\sqrt{\frac{9\beta}{2\pi t}}\\ &\times\left\langle\lambda(1-\beta)u'(0)-\lambda(1-\beta)G_i+c_1\frac{w_{i0}}{2t}\right\rangle\\ &+\lambda^2(1-\beta)c_1G_i+\lambda^2w_{i0}\frac{9\beta\Delta_H^2}{2}\delta(t)+\lambda\Delta_H\sqrt{\frac{9\beta}{2\pi}}w_{i0}\delta^2(t),\end{aligned}\qquad[3]$$

with the following initial conditions for the velocity,

$$w_i(0)=w_{i0},\qquad[4a]$$

and for the acceleration (this is obtained directly from [1]),

$$\frac{dw_i}{dt}(0)=-\lambda c_1w_{i0}-\lambda(1-\beta)u_i'(0)+\lambda(1-\beta)G_i-w_{i0}\lambda\Delta_H\sqrt{\frac{9\beta}{2\pi}}\delta(t).\qquad[4b]$$

The Dirac delta appears in the above equations always in conjunction with the initial relative velocity $w_{i0}$ only. This is a manifestation of the fact that if the particle is introduced in the flow with a finite relative velocity at time $t=0$, then an impulse acts upon it as a result of the fluid's reaction. In the case of zero initial relative velocity, which is of interest in most practical cases, the transformed equation becomes:

$$\frac{\mathrm{d}^2 w_i}{\mathrm{d}t^2} + \lambda\left(2c_1 - \frac{9\beta\lambda\Delta_H^2}{2}\right)\frac{\mathrm{d}w_i}{\mathrm{d}t} + \lambda^2 c_1^2 w_i = -\lambda(1-\beta)\frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} - \lambda^2(1-\beta)c_1\frac{\mathrm{d}u_i}{\mathrm{d}t}$$
$$+\lambda^2(1-\beta)\Delta_H\sqrt{\frac{9\beta}{2\pi}}\int_0^t \frac{\frac{\mathrm{d}^2 u_i}{\mathrm{d}\sigma^2}}{(t-\sigma)^{0.5}}\mathrm{d}\sigma + \lambda^2\Delta_H\sqrt{\frac{9\beta}{2\pi t}}$$
$$\times\langle(1-\beta)u'(0)-(1-\beta)G_i\rangle + \lambda^2 c_1(1-\beta)G_i. \qquad [5]$$

Equations [3] and [5] are explicit in $w_i$. Their numerical solution may be obtained by any standard numerical solution technique, explicit or implicit, and normally does not require iterations. They also have the additional advantage that the history integral term contains the second derivative of the fluid velocity only. Therefore, their solution requires less computational memory than [1]. The disadvantage of the equations is that they are second-order differential equations and that they contain more terms than [1]. However, the numerical advantages by far outweigh the disadvantages. When computations for particle trajectories were made in a sinusoidal flow field (Michaelides 1992), the transformed equation resulted in a CPU time reduction from 6- to 11-fold.

While transforming [1] it appears that the coefficients $c_1$, $\Delta_A$ and $\Delta_H$ have been treated as constants. This procedure, although it appears not to be mathematically rigorous, is logically equivalent to the one that enables us to use these empirical coefficients in the first place: one first derives the governing equation at the limit of low Reynolds numbers and subsequently multiplies the resulting terms by the respective empirical coefficients. Similarly in the present method, the transformed equation of motion is first derived at the low Reynolds number limit and subsequently the various terms arising are multiplied by their respective empirical coefficients. This approximation enables the transformation, which is impossible to achieve otherwise.

## SOLUTION OF THE EQUATION—DOES THE HISTORY TERM MATTER?

Having developed an explicit form (in particle velocity) of the equation of motion of the particle, it is now easy to perform computations for particle velocities and trajectories in any (known) unsteady flow field. The question that needs to be answered is "Under what conditions does the history term in this equation make a substantial contribution to the desired results?" and also whether time-integral quantities, such as particle position or dispersion, are substantially affected. In the cases where the history term does not play an important role one may neglect it and use the simpler form of the first-order differential equation ([1] without the history term), which is easier and more convenient to solve.

The equation of motion of the particle is solved for the following three cases: (a) the case of sinusoidal fluid flow, where the response of the particle is followed for one-half of the cycle; (b) the case of a random velocity field superimposed on a uniform flow; and (c) the case of fluid velocity, which increases monotonically in steps of random magnitude and duration. Case (c) is obtained from case (b) by taking the absolute values of the same random numbers and adding them to the previous value of the fluid velocity.

Throughout the calculations the following empirical equations were used for the three coefficients $c_1$, $\Delta_A$ and $\Delta_H$:

$$c_1 = 1 + 0.15\,\mathrm{Re}_P^{0.667}, \quad \Delta_A = 2.1 - \frac{0.132\,\mathrm{Ac}^2}{(1+0.12\,\mathrm{Ac}^2)} \quad \text{and} \quad \Delta_H = 0.48 + \frac{0.5\,\mathrm{Ac}^3}{(1+\mathrm{Ac})^3}, \qquad [6a]$$

where $\mathrm{Re}_P$ is the particle Reynolds number and Ac is the acceleration number. In terms of the dimensionless velocity and acceleration and the other quantities defined above, these numbers are as follows:

$$\mathrm{Re}_P = \frac{2\alpha|U_0 w_i|\rho_F}{\mu} \quad \text{and} \quad \mathrm{Ac} = \frac{18\beta}{\mathrm{Re}_P}\left|\frac{\mathrm{d}w_i}{\mathrm{d}t}\right|. \qquad [6b]$$

The drag coefficient is the one which is most often used in particle flow simulations. For the other coefficients we have used, for demonstration purposes, the expressions derived by Odar & Hamilton (1964). These expressions are most frequently used and have been verified by Tsuji *et al.* (1991) for particle flows at higher Reynolds number.

In the sinusoidal case, where the particle is driven by a sinusoidal flow velocity $u_i = \sin(\omega t)$, the particle velocity is calculated with and without the history term. Figure 1a shows the fluid and particle velocities after one-half cycle for a dimensionless frequency $\omega = 10$ (the frequency is made

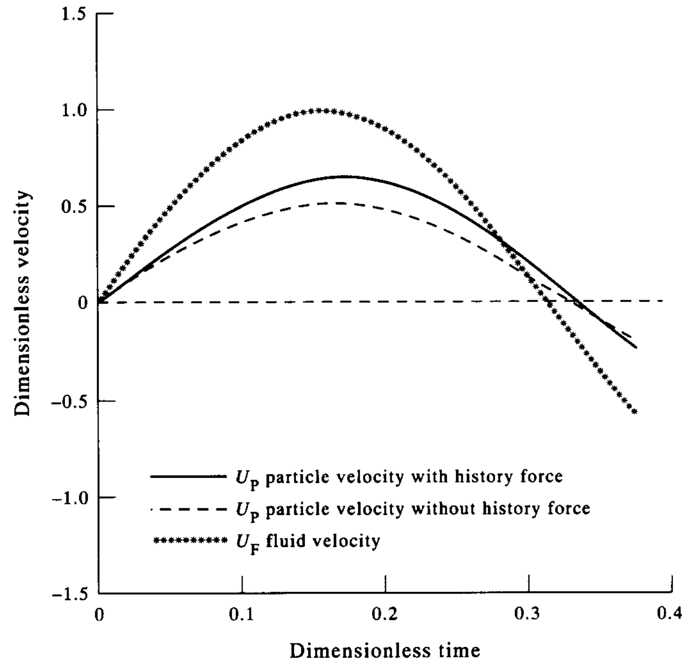


Figure 1a. Response of a particle to the sinusoidal fluid velocity with and without the history term ($\omega = 10$, $\beta = 1/2.7$).

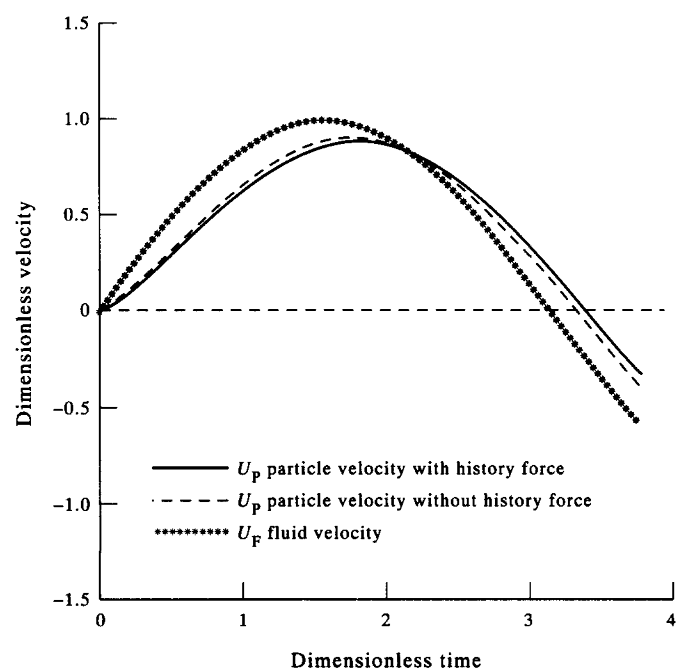


Figure 1b. Response of a particle to the sinusoidal fluid velocity with and without the history term ($\omega = 1$, $\beta = 1/2.7$, symbols as in figure 1a).

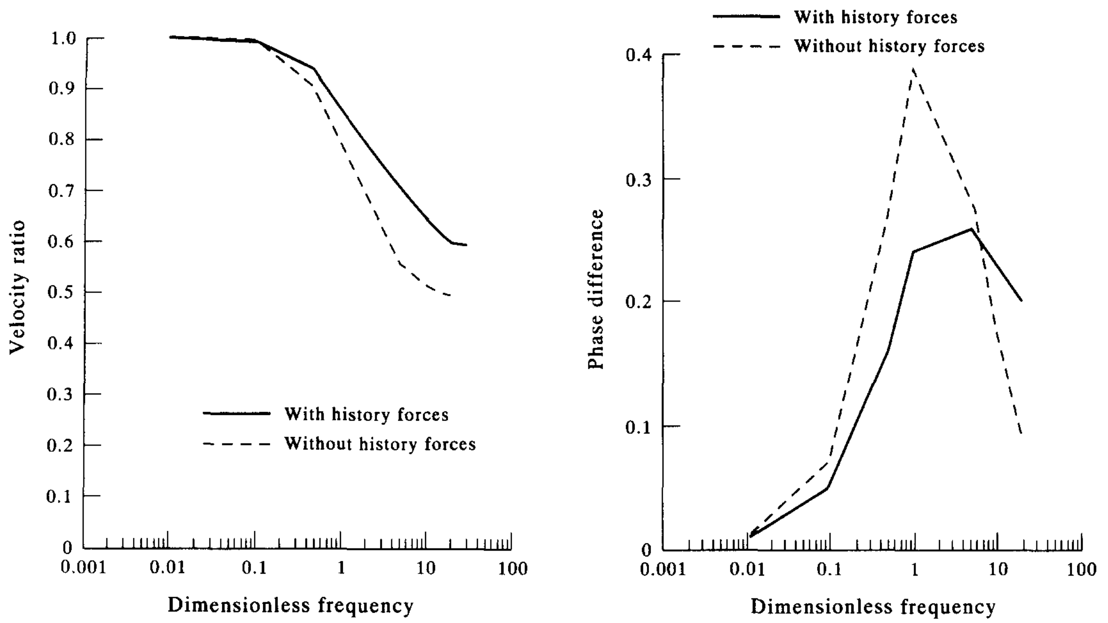


Figure 2a. Velocity ratio $\eta$ vs the dimensionless frequency $\omega$ ($\beta = 1/2.7$).

Figure 2b. Phase difference $\phi$ vs the dimensionless frequency $\omega$ ($\beta = 1/2.7$).

dimensionless by multiplying by the characteristic time of the particle). Figure 1b shows the response of the particle when $\omega = 1$. Both figures correspond to $\beta = 1/2.7$, which is approximately equal to the density ratio in water–sand mixtures. It is evident that the particle lags the motion of the fluid and that the amplitude of its velocity is lower. It is also seen that the particle velocity is affected more in the higher frequency case.

It is apparent from figures 1a and 1b that the inclusion of the history term in the equation of motion of the particle has an effect on both the amplitude of the velocity and the phase lag of the particle with respect to the fluid. We may define an amplitude ratio $\eta$ and a phase difference $\phi$ for the velocities of the particle and the fluid and calculate these quantities with and without the history term. Figure 2a shows the velocity ratio $\eta$ as a function of the dimensionless frequency for $\beta = 1/2.7$ and figure 2b shows the phase difference between the fluid and the particle for the same $\beta$. The frequency is made dimensionless by multiplying by the characteristic time of the particle. It is seen in both figures that the higher discrepancies in the results with and without the history term occur at higher frequencies. Neglecting the history term at the higher frequencies would result in the underprediction of the velocity ratio by 15–20%.

The effect of the density ratio $\beta$ was found to be of importance in the range $0.7 > \beta > 0.002$. Figure 3 shows the dependence of the velocity ratio $\eta$ on $\beta$ at the dimensionless frequency of 10. Even at this rather high frequency it is apparent that neglecting the history term makes very little difference in gas–solid flows. However, in liquid–solid flows (where $\beta$ is close to 0.3) neglecting the history term may result in errors of the order of 20–30%, even at moderate frequencies.

Calculations were also conducted with the velocity of the fluid being equal to $1 + u'$, where $u'$ is a random velocity field. This random velocity field has a standard deviation 0.2 and each random component acts on the particle for a random but positive time interval $\Delta t$. Figure 4 shows the particle velocity with and without the history term for 10 time intervals. It is evident that when the fluid velocity is randomly fluctuating around a given value the particle velocity with the history term sometimes is less and sometimes more than the particle velocity without the history term. When one integrates these velocities to obtain average velocities or particle trajectories (and hence particle dispersion), the alternating differences cancel. Therefore, one expects very small differences in the final position of the particle after a large number of velocity steps. This was actually observed when the calculations proceeded to 100 time intervals: regardless of the standard deviation of the random time intervals, the final positions of the particle, calculated with and without the history term, were $< 1\%$ from each other. This means that neglecting the history term has insignificant impact on calculations for particle dispersion or average particle velocity.

The effect of the history term was found to be significant on particle disperison when the fluid velocity varied monotonically. This variation was simulated by adding to the instantaneous fluid velocity the absolute value of a random number acting after an interval $\Delta t$. Figure 5 shows the fluid velocity field and the resulting particle velocity, calculated with and without the history term. In the calculations without the history term the particle velocity lagged considerably when the time intervals were short. Because of this, particles traveled faster and further when the history term was included. The discrepancy is manifested in the distance particles traveled after 100 time intervals. Figure 6 shows the fractional difference in the distance traveled as a function of the standard deviation of the time intervals. The inverse of the latter is an approximate measure of the frequency of variation of the velocity field. It is seen that the discrepancy of the calculations with and without the history term becomes significant at low standard deviations of time (which correspond to high dimensionless frequencies of the velocity field).

The results depicted in the figures 4–6 show that the effect of the history term can be justifiably neglected in all the cases when average results are sought resulting from a random fluid velocity. However, if the fluid velocity varies monotonically (e.g. in nozzle flow or a jet) the history term may only be neglected if the dimensionless frequency of variation of the velocity field is relatively low.

It must be emphasized that the effect of the particle size is included in the dimensionless times and frequencies. These quantities are made dimensionless by using the characteristic time of the particle $\tau_P = 2\rho_P\alpha^2/9\mu$. Thus, a dimensionless frequency $\omega$ corresponds to a real frequency of

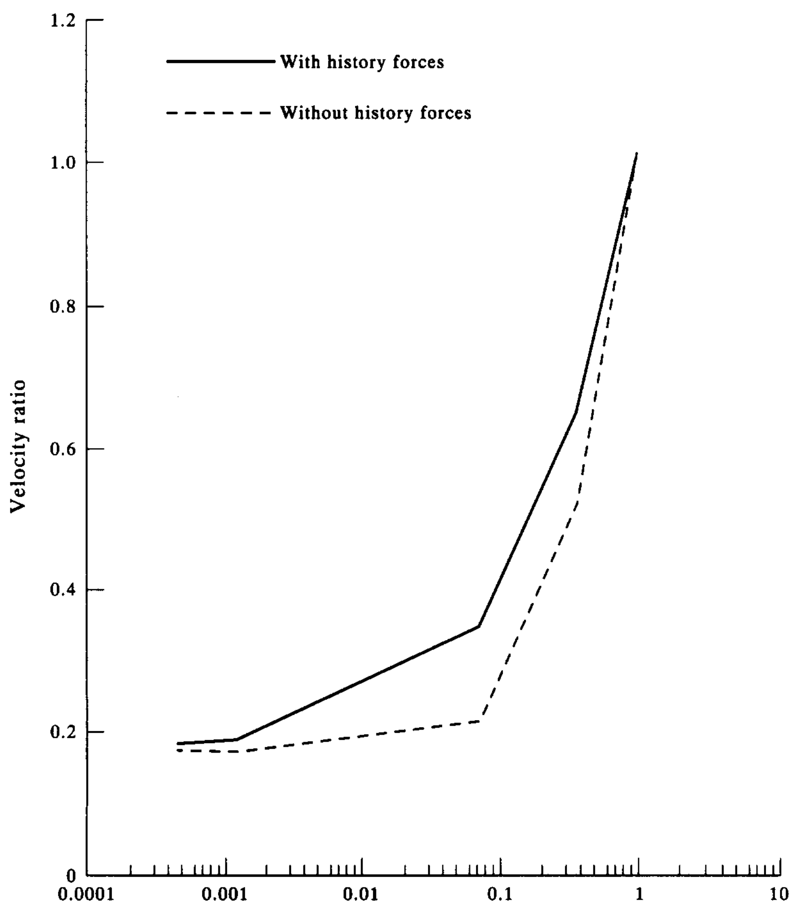


Figure 3. Velocity ratio $\eta$ vs the density ratio $\beta$ ($\omega = 10$).

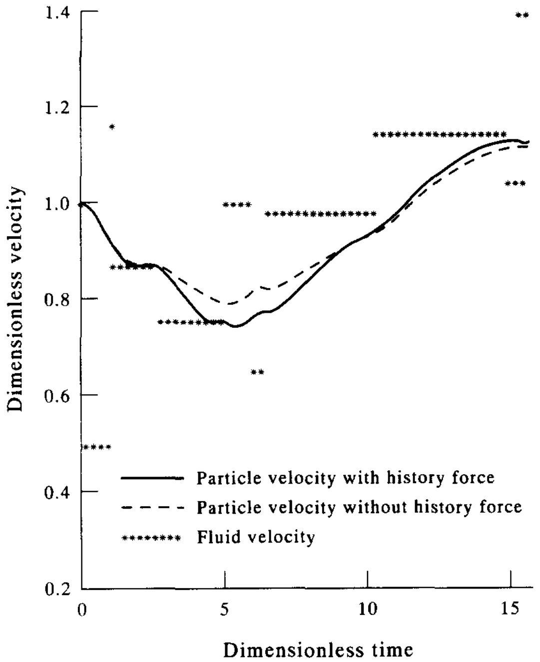


Figure 4. Response of a particle to random fluid velocity fluctuations with and without the history term ($\beta = 1/2.7$).

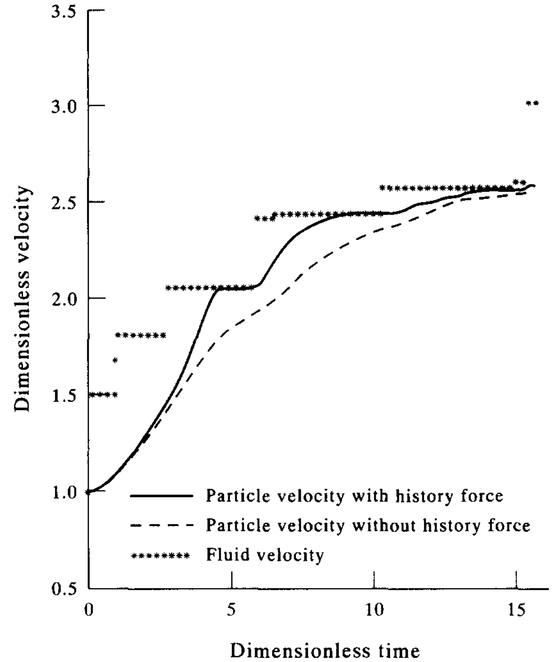


Figure 5. Response of a particle to monotonically increasing fluid velocity with and without the history term ($\beta = 1/2.7$).

$\omega_r = 9\omega\mu/(2\rho_P\alpha^2)$. Hence, doubling the particle size quadruples the flow frequency, for which the effects of the history term become significant. Consequently, the effect of the history term is considerably more significant in calculations involving fine particles than coarse particles, other parameters being equal.

## AN HISTORICAL NOTE

It is common to encounter in the literature reference to the history term in the equation of motion of the particle as the "Basset force" or the "Basset term". This is because the first introduction of the term is frequently attributed to the British hydrodynamicist A. B. Basset (Basset 1888). Indeed, in Basset's *A Treatise on Hydrodynamics* (Vol. 2, p. 291), the term appears with the other forces acting on a particle moving in a viscous fluid. Basset derived this integral term from the transient hydrodynamic reaction of the fluid to the motion of the sphere (p. 289). However, the French physicist J. Boussinesq had already published (in 1885) an article where the same integral term appears explicitly in the expression of the fluid resistance to the motion of the particle (Boussinesq 1885a). The equation for the fluid resistance derived by Boussinesq is in the form most commonly used today. Boussinesq derived this term and the equation for the resistance of the sphere based on a general method for the solution of equations of potentials in spherical coordinates. This solution is exposed in his book on the applications of potentials, also published in 1885 (Boussinesq 1885b). In this book (pp. 413–416) Boussinesq also points out the presence of similar history integrals in several heat transfer processes. Boussinesq's paper and book represent a thorough derivation of the equation of motion of a particle and leave no doubt of his precedence regarding the discovery of the history term.

It is, therefore, evident that Boussinesq has precedence over Basset in the derivation and introduction of the history term in the equation of motion of a sphere. If this term is to be named after someone, it is neither appropriate nor fair to name it after the second person who derived it. For this reason it should be called either the "Boussinesq–Basset" term or simply the "history" term.

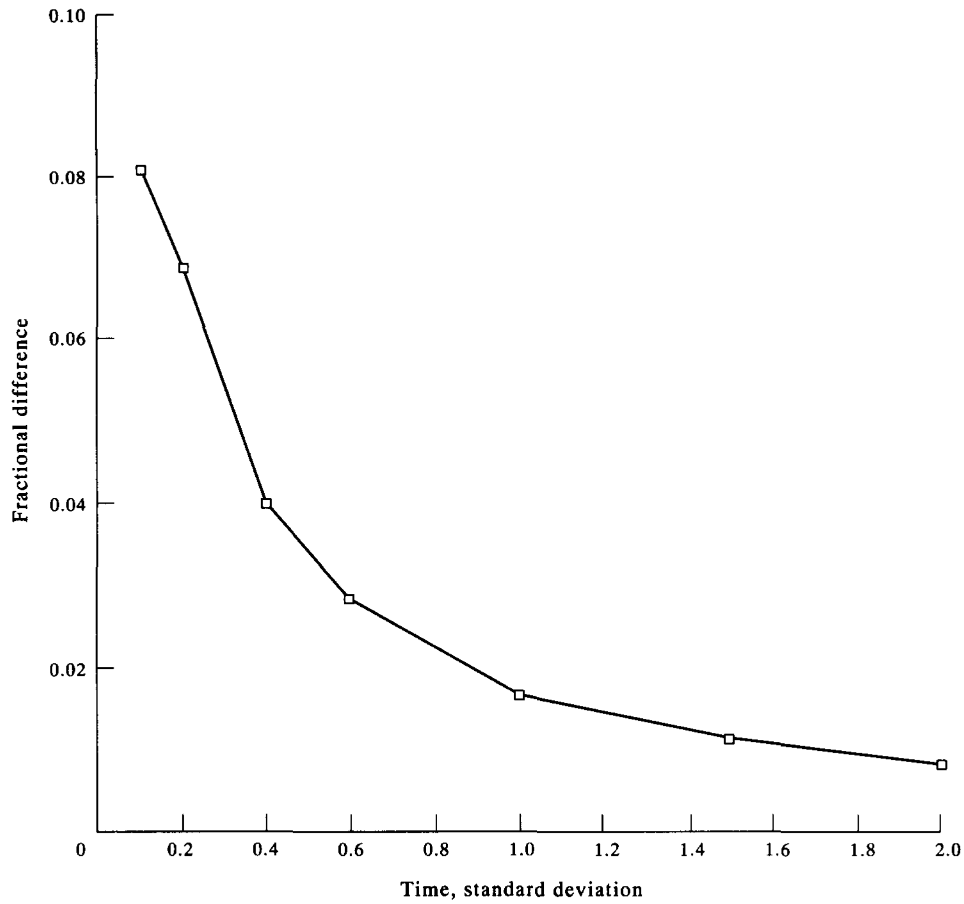


Figure 6. Effect of the standard deviation of time steps on the distance traveled by the particle after 100 steps ($\beta = 1/2.7$).

## CONCLUSIONS

By using an integrodifferential operator the full equation of motion of the particle is transformed to a second-order ordinary differential equation which is explicit in the particle velocity. Empirical coefficients for the drag, history and added mass terms account for deviations from the creeping flow assumption. Since the transformed equation is explicit in the particle velocity, it is easier and computationally more efficient to perform calculations with it. Calculations were performed for three cases of fluid velocities: (a) sinusoidal; (b) random superposition on a uniform velocity value; and (c) monotonically increasing by random steps. It was found that the history term can be neglected in the following cases: (a) random fluid velocity field, if one is interested in time-average results or integral quantities; (b) the fluid-to-particle density ratio is $< 0.002$, which corresponds to most gas–solid flows; and (c) the dimensionless frequency of variation is $< 0.5$. On the other hand, it was observed that the history term is important in the calculations if the dimensionless frequency of velocity fluctuations is $> 0.5$ in the sinusoidal case and $> 2$ in the monotonically increasing case (even if only average results are of interest). The effect of the history term is more pronounced when $\beta$ is in the range 0.7–0.002, which corresponds to liquid–solid flows.

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